

# Minimal surfaces in $q$ -deformed $\text{AdS}_5 \times \text{S}^5$ with Poincare coordinates

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## Abstract

We study minimal surfaces in  $q$ -deformed  $\text{AdS}_5 \times \text{S}^5$  with a new coordinate system introduced in the previous work 1408.2189. In this letter, we introduce Poincare coordinates for the deformed theory. Then we construct minimal surfaces whose boundary shape is a circle. The solution corresponds to a  $1/2$  BPS circular Wilson loop in the  $q \rightarrow 1$  limit. A remarkable point is that the classical Euclidean action is not divergent unlike the original one. This finiteness indicates that the  $q$ -deformation may be regarded as a UV regularization.

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# 1 Introduction

The AdS/CFT correspondence [1] is a realization of the equivalence between string theories and gauge theories. The most well-studied example is the duality between type IIB string theory on  $\text{AdS}_5 \times S^5$  and  $\mathcal{N} = 4$  super Yang-Mills theory. It is often likened to a “harmonic oscillator” among the gauge/gravity dualities. A great discovery in the recent is an integrable structure behind the AdS/CFT [2, 3]. The integrability plays an important role in testing the conjectured relations in the AdS/CFT. In particular, type IIB superstring theory on  $\text{AdS}_5 \times S^5$  is realized as a supersymmetric coset sigma model [4] and its classical integrability is discussed in [5].

The integrable structure behind the AdS/CFT opens up a new arena for studying integrable deformations as well. It would be of significance to reveal a deeper integrable structure behind gauge/gravity dualities beyond the AdS/CFT. To tackle this issue, it is desirable to employ systematic ways to consider integrable deformations of two-dimensional non-linear sigma models. One way is to follow the Yang-Baxter sigma model approach [6]. It has recently attracted a great deal of attention. According to this approach, an integrable deformation is fixed by picking up a skew-symmetric classical  $r$ -matrix satisfying the modified classical Yang-Baxter equation (mCYBE).

It was originally proposed for principal chiral models [6] and generalized to symmetric cosets [7]<sup>1</sup>. Based on this generalization, the classical action of a  $q$ -deformed  $\text{AdS}_5 \times S^5$  superstring was constructed in [12]. Then the metric<sup>2</sup> and NS-NS two-form have been fixed in [13], although, the dilaton and R-R sector have not been determined yet. In particular, a singular surface exists in the  $q$ -deformed  $\text{AdS}_5$ . For this deformed string theory, many works have been done. A certain limit of the deformed  $\text{AdS}_n \times S^n$  has been studied in [14]. A mirror description was proposed in [15]. The fast-moving string limit and the corresponding spin chain were studied in [16]. Giant magnon solutions have been studied in [15, 17–19]. Deformed Neumann models were also derived in [20]. A new coordinate system to study the holographic relation was introduced in [21]. For  $q$ -deformations at root of unity, see [22, 23].

There exists another kind of integrable deformations based on the classical Yang-Baxter equation (CYBE), rather than the mCYBE. This approach is closely related to Jordanian

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<sup>1</sup> For earlier developments on  $q$ -deformations of  $su(2)$  and  $sl(2)$ , see [8–11].

<sup>2</sup>We refer to it as the ABF metric.

deformations of the  $\text{AdS}_5 \times \text{S}^5$  superstring [24]. Classical  $r$ -matrices found in [25–28] lead to type IIB supergravity solutions including  $\beta$ -deformed backgrounds [29] and gravity duals for non-commutative gauge theories [30]. The correspondence of this type is referred as the gravity/CYBE correspondence (For a short summary, see [31]).

In this letter, we are interested in the  $q$ -deformed  $\text{AdS}_5 \times \text{S}^5$  superstring [12]. An interesting issue is to consider a holographic relation in the  $q$ -deformed background. A proposal is that the singularity surface in the ABF metric [13] may be regarded as the holographic screen [21]. Thus, it is nice to introduce a new coordinate system which describes only the spacetime enclosed by the singularity surface [21]. With this coordinate system, minimal surfaces whose boundary is a straight line have been considered in [21]. The solutions are reduced to the well-known results [32, 33] in the  $q \rightarrow 1$  limit. It is quite suggestive that the relations of the conserved charges satisfy the standard results in the usual AdS/CFT even though the supersymmetry is also  $q$ -deformed.

We will here continue to study minimal surfaces in the Euclidean  $q$ -deformed AdS space by introducing the Poincare coordinates. Then we construct minimal surfaces whose boundary shape is a circle. A solution corresponds to a 1/2 BPS circular Wilson loop in the undeformed limit. A remarkable point is that the classical Euclidean action is not divergent unlike the original one. The finiteness indicates that the  $q$ -deformation may be regarded as a UV regularization.

This letter is organized as follows. Section 2 explains a new coordinate system for the  $q$ -deformed  $\text{AdS}_5 \times \text{S}^5$  background. Then we introduce the associated Poincare coordinate system for the deformed AdS part. In section 3, we consider two types of minimal surfaces which end up with a circle at the boundary. The former is constructed by supposing that the surface extends only within the deformed  $\text{AdS}_2$  subspace. The latter is a generalization of the former including a polar angle in the deformed  $\text{S}^2$ . The solutions corresponds to a 1/2 BPS circular Wilson loop [34, 35] and a 1/4 BPS one [36], respectively, in the undeformed limit  $q \rightarrow 1$ . A remarkable point is that in both cases, the resulting classical action is finite even though there is no UV cut-off. In section 4, we consider a minimal surface with a cylinder shape. The resulting classical action is not divergent unlike the undeformed one. Section 5 is devoted to conclusion and discussion. Appendix A argues space-like geodesics to the singularity surface in the deformed AdS. In Appendix B, we introduce the Poincare coordinates for the ABF metric [13].

## 2 A $q$ -deformation of $\text{AdS}_5 \times \text{S}^5$

We consider the bosonic part of the classical action of a  $q$ -deformed  $\text{AdS}_5 \times \text{S}^5$  superstring [12]. With the conformal gauge, the bosonic action (in the string frame) is composed of the metric part and the Wess-Zumino (WZ) term.

In the following, we are concerned with minimal surfaces ending at the singularity surface in the coordinate system introduced in [13]. It is helpful to employ a new coordinate system which describes the spacetime only inside the singularity surface [21]. In the new coordinates, the singularity surface is located at the boundary.

Let us first introduce the metric part and the WZ term of the bosonic action with the new coordinate system [21]. Then we introduce the associated Poincare coordinates.

### 2.1 The bosonic part of the $q$ -deformed action

We consider the bosonic part of the classical action of the  $q$ -deformed  $\text{AdS}_5 \times \text{S}^5$  superstring.

The bosonic action (in the conformal gauge) is composed of the metric part  $S_G$  and the Wess-Zumino (WZ) term  $S_{\text{WZ}}$  like

$$S = S_G + S_{\text{WZ}}.$$

Here  $S_{\text{WZ}}$  describes the coupling of string to an NS-NS two-form.

The metric part  $S_G$  is divided into the deformed AdS and internal sphere parts;

$$\begin{aligned} S_G &= \int d\tau d\sigma [\mathcal{L}_{\text{AdS}}^G + \mathcal{L}_{\text{S}}^G] \\ &= -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \eta_{\mu\nu} \left[ G_{\text{AdS}}^{MN} \partial^\mu X_M \partial^\nu X_N + G_{\text{S}}^{PQ} \partial^\mu Y_P \partial^\nu Y_Q \right], \end{aligned}$$

where the string world-sheet coordinates are  $\sigma^\mu = (\tau, \sigma)$  with  $\eta_{\mu\nu} = (-1, +1)$ .

With a new coordinate system proposed in [21], the metric for the deformed AdS and sphere parts are given by, respectively,

$$\begin{aligned} ds_{\text{AdS}_5}^2 &= R^2 \sqrt{1 + C^2} \left[ -\cosh^2 \chi dt^2 + \frac{d\chi^2}{1 + C^2 \cosh^2 \chi} \right. \\ &\quad \left. + \frac{(1 + C^2 \cosh^2 \chi) \sinh^2 \chi}{(1 + C^2 \cosh^2 \chi)^2 + C^2 \sinh^4 \chi \sin^2 \zeta} (d\zeta^2 + \cos^2 \zeta d\psi_1^2) + \frac{\sinh^2 \chi \sin^2 \zeta d\psi_2^2}{1 + C^2 \cosh^2 \chi} \right], \end{aligned} \quad (2.1)$$

$$ds_{S^5}^2 = R^2 \sqrt{1+C^2} \left[ \cos^2 \gamma d\phi^2 + \frac{d\gamma^2}{1+C^2 \cos^2 \gamma} + \frac{(1+C^2 \cos^2 \gamma) \sin^2 \gamma}{(1+C^2 \cos^2 \gamma)^2 + C^2 \sin^4 \gamma \sin^2 \xi} (d\xi^2 + \cos^2 \xi d\phi_1^2) + \frac{\sin^2 \gamma \sin^2 \xi d\phi_2^2}{1+C^2 \cos^2 \gamma} \right]. \quad (2.2)$$

Here the deformed  $\text{AdS}_5$  is parameterized by the coordinates  $(t, \psi_1, \psi_2, \zeta, \chi)$ , while the deformed  $S^5$  is described by  $(\phi, \phi_1, \phi_2, \xi, \gamma)$ . The deformation is characterized by a real parameter  $C \in [0, \infty)$ . When  $C = 0$ , the geometry is reduced to the undeformed  $\text{AdS}_5 \times S^5$  with the curvature radius  $R$ . A relation between the ABF coordinates and the new ones is depicted in Fig.1. It should be mentioned that a curvature singularity exists at  $\chi = \infty$  in

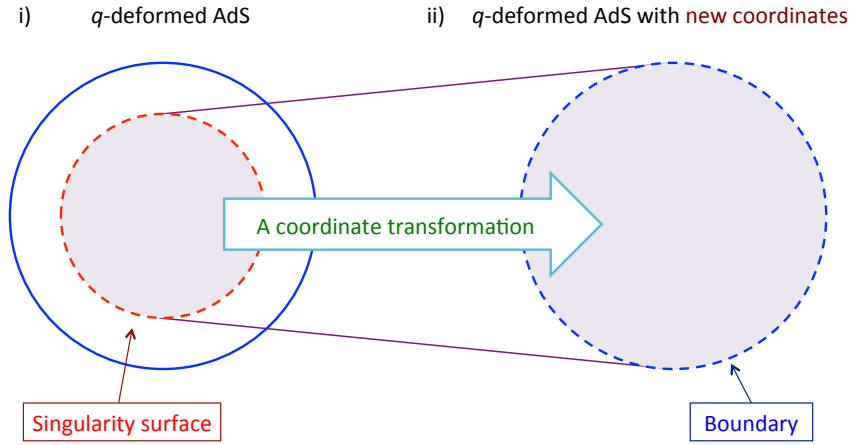


Figure 1: A new coordinate system for the  $q$ -deformed  $\text{AdS}_5$ . The resulting geometry is enclosed by the singularity surface.

the new coordinate system as well. In [21], it is shown that the proper time from any point to the singularity is infinite, while it does not take infinite time to reach the singularity in the coordinate time. This is the same result as in the usual  $\text{AdS}$  space with the global coordinates. Thus it seems likely to regard the singularity surface as the boundary in the holographic setup for the  $q$ -deformed geometry.

The WZ term  $S_{\text{WZ}}$  for the  $\text{AdS}$  part and the sphere part are given by, respectively,

$$\mathcal{L}_{\text{AdS}}^{\text{WZ}} = \frac{T_{(C)}}{2} C \epsilon^{\mu\nu} \frac{\sinh^4 \chi \sin 2\zeta}{(1+C^2 \cosh^2 \chi)^2 + C^2 \sinh^4 \chi \sin^2 \zeta} \partial_\mu \psi_1 \partial_\nu \zeta, \quad (2.3)$$

$$\mathcal{L}_S^{\text{WZ}} = -\frac{T_{(C)}}{2} C \epsilon^{\mu\nu} \frac{\sin^4 \gamma \sin 2\xi}{(1+C^2 \cos^2 \gamma)^2 + C^2 \sin^4 \gamma \sin^2 \xi} \partial_\mu \phi_1 \partial_\nu \xi. \quad (2.4)$$

Here the totally anti-symmetric tensor  $\epsilon^{\mu\nu}$  is normalized as  $\epsilon^{01} = +1$ . We have introduced a  $C$ -dependent string tension  $T_{(C)}$  defined as

$$T_{(C)} \equiv T\sqrt{1+C^2}, \quad T \equiv \frac{R^2}{2\pi\alpha'}, \quad \sqrt{\lambda} \equiv \frac{R^2}{\alpha'}. \quad (2.5)$$

Each component of the WZ term is proportional to  $C$ , and hence it vanishes when  $C = 0$ .

## 2.2 Poincare coordinates

So far, we have discussed in the global coordinates. However, for our purpose, it is desirable to introduce the Poincare coordinates for the deformed AdS part (2.1).

Let us first perform the following coordinate transformation

$$\cosh \chi \equiv \frac{1}{\cos \theta}, \quad (2.6)$$

and the Wick rotation  $t \rightarrow i\tau$  to move to the Poincare coordinates. The metric (2.1) is rewritten as the deformed EuclideanAdS<sub>5</sub>,

$$ds_{\text{AdS}_5}^2 = R^2\sqrt{1+C^2} \left[ \frac{d\tau^2}{\cos^2 \theta} + \frac{d\theta^2}{\cos^2 \theta + C^2} + \frac{(\cos^2 \theta + C^2) \sin^2 \theta}{(\cos^2 \theta + C^2)^2 + C^2 \sin^4 \theta \sin^2 \zeta} (d\zeta^2 + \cos^2 \zeta d\psi_1^2) + \frac{\sin^2 \theta \sin^2 \zeta d\psi_2^2}{\cos^2 \theta + C^2} \right]. \quad (2.7)$$

Then, performing a coordinate transformation:

$$z \equiv e^\tau \cos \theta, \quad r \equiv e^\tau \sin \theta, \quad (2.8)$$

we can obtain the following metric,

$$ds_{\text{AdS}_5}^2 = R^2\sqrt{1+C^2} \left[ \frac{dz^2 + dr^2}{z^2 + C^2(z^2 + r^2)} + \frac{C^2(z dz + r dr)^2}{z^2(z^2 + C^2(z^2 + r^2))} + \frac{(z^2 + C^2(z^2 + r^2))r^2}{(z^2 + C^2(z^2 + r^2))^2 + C^2r^4 \sin^2 \zeta} (d\zeta^2 + \cos^2 \zeta d\psi_1^2) + \frac{r^2 \sin^2 \zeta d\psi_2^2}{z^2 + C^2(z^2 + r^2)} \right]. \quad (2.9)$$

When  $C = 0$ , one can reproduce the Euclidean AdS<sub>5</sub> metric with the Poincare coordinates. In this sense, the metric (2.9) may be regarded as the Poincare version of the deformed AdS<sub>5</sub>. Note that the Lorentzian signature can usually be recovered via the relation  $dr^2 + r^2 d\Omega_3^2 = dt_P^2 + \sum_{i=1}^3 dx_i^2$ , where  $t_P$  is the (Euclidean) Poincare time, but for the deformed AdS geometry above it seems difficult to move back to the Lorentzian signature directly.

After the Wick rotation has been performed, all we are concerned with are space-like paths rather than time-like and null ones. As argued in App. A, the space-like proper distance to the singularity surface is finite when  $C \neq 0$ , in comparison to the undeformed case with  $C = 0$ . This property will be crucial in the next section.

### 3 Minimal surfaces

Let us study minimal surfaces in the  $q$ -deformed  $\text{AdS}_5 \times \text{S}^5$  string with the Poincare coordinates. Note that we are working in the Euclidean signature and also on the string world-sheet. In particular, we consider minimal surfaces whose boundary shape is a circle for two cases, 1) the  $\text{AdS}_2$  subspace 2) the  $\text{AdS}_2 \times \text{S}^2$  subspace. In both cases, for non-vanishing  $C$ , the classical actions are finite without introducing a UV cut-off. In the  $q \rightarrow 1$  limit, the solutions are reduced to minimal surfaces dual for a 1/2 BPS circular Wilson loop [34, 35] and a 1/4 BPS circular Wilson loop [36], respectively.

#### 3.1 A deformed $\text{AdS}_2$ subspace

We shall consider minimal surfaces which ends up with a circle at the boundary of the  $q$ -deformed  $\text{AdS}_5$  with the Poincare coordinates (2.9). Let us consider an ansatz with the conformal gauge:

$$z = \sqrt{a^2 - r^2}, \quad r = r(\sigma), \quad \psi_1 = \psi_1(\tau), \quad \psi_2 = \zeta = 0, \quad (3.1)$$

with  $0 \leq \tau < 2\pi$ ,  $0 \leq \sigma < \infty$ . Here  $a$  is the radius of the circle at the boundary. Note that (3.1) is a consistent ansatz and the WZ term vanishes under (3.1) as explained in [14]. Then the geometry with the metric (2.9) is reduced to the following deformed  $\text{AdS}_2$ ,

$$ds_{\text{AdS}_2}^2 = \frac{R^2 \sqrt{1 + C^2} r^2}{(1 + C^2) a^2 - r^2} \left[ \frac{a^2 dr^2}{r^2 (a^2 - r^2)} + d\psi_1^2 \right]. \quad (3.2)$$

This metric leads to a solution of the string equation of motion:

$$z = a \tanh \sigma, \quad r = \frac{a}{\cosh \sigma}, \quad \psi_1 = \tau. \quad (3.3)$$

The next task is to evaluate the classical Euclidean action of this solution. By following the undeformed case, let us formally introduce a cut-off  $\epsilon$  for the coordinate  $z$ . It may be

regarded as a cut-off  $\sigma_0$  for the string world-sheet coordinate  $\sigma$  through the classical solution (3.3) for  $z$ ,

$$\epsilon = a \tanh \sigma_0. \quad (3.4)$$

Then the classical action is evaluated as

$$\begin{aligned} S &= \frac{\sqrt{\lambda}}{4\pi} \sqrt{1+C^2} \int_0^{2\pi} d\tau \int_{\sigma_0}^{\infty} d\sigma \left[ \frac{2}{\sinh^2 \sigma + C^2 \cosh^2 \sigma} \right] \\ &= \sqrt{\lambda} \frac{\sqrt{1+C^2}}{C} \left( \operatorname{arccot}[C] - \operatorname{arccot}\left[\frac{Ca}{\epsilon}\right] \right), \end{aligned} \quad (3.5)$$

where  $\lambda \equiv R^4/\alpha'^2$ . Note that the second term in (3.5) can be expanded in terms of  $\epsilon$  like

$$-\sqrt{\lambda} \frac{\sqrt{1+C^2}}{C} \operatorname{arccot}\left[\frac{Ca}{\epsilon}\right] = -\sqrt{\lambda} \frac{\sqrt{1+C^2}}{C} \left( \frac{\epsilon}{Ca} + \mathcal{O}(\epsilon^2) \right), \quad (3.6)$$

where  $C$  has been fixed in this expansion. One can readily see that the cut-off can be removed for non-vanishing  $C$ . By taking  $\epsilon \rightarrow 0$  the classical action (3.5) becomes,

$$S = \sqrt{\lambda} \frac{\sqrt{1+C^2}}{C} \operatorname{arccot}[C]. \quad (3.7)$$

It should be remarked that the action (3.7) is finite even if there is no UV cut-off for the string world-sheet (equivalently for the radial direction of the deformed AdS). The result would come from the finiteness of the space-like proper distance to the singularity surface (See App. A). Then the deformation parameter  $C$  works as a UV regularization and one does not need to introduce  $\epsilon$  any more in evaluating the classical action. This point becomes clear by taking the  $C \rightarrow 0$  limit of (3.7). By expanding (3.7) in terms of  $C$ , we obtain the following expression:

$$S = -\sqrt{\lambda} + \sqrt{\lambda} \frac{\pi}{2C} + \mathcal{O}(C). \quad (3.8)$$

However we need to include  $\epsilon$  so as to reproduce the regularized result in the undeformed limit [34, 35] as shown below. For this purpose, it is helpful to consider the undeformed limit of the classical action (3.5) by taking  $C \rightarrow 0$  while keeping  $\epsilon$  finite. This corresponds to keeping the boundary of the solution away from the singularity surface. Then the result in the undeformed case [34, 35] is reproduced,

$$S = -\sqrt{\lambda} + \sqrt{\lambda} \frac{a}{\epsilon}. \quad (3.9)$$



Note that the divergent term is also reproduced in the undeformed limit and it can be canceled by taking account of a Legendre transformation as usual.

Note also that the action (3.7) is independent of  $a$  due to the scale invariance of the metric (2.9) under  $z \rightarrow c_0 z$  and  $r \rightarrow c_0 r$  where  $c_0$  is a positive real constant.

It may be nice to write the classical action (3.7) with the  $q$ -deformation parameter. The deformation parameter  $C$  is related to the  $q$ -deformation parameter through the following relation [12, 13],

$$q = e^{-\nu/T}, \quad \nu = \frac{C}{\sqrt{1+C^2}}. \quad (3.10)$$

Then the result (3.7) is expressed as

$$S = \frac{\sqrt{\lambda}}{\nu} \operatorname{arccot} \left[ \frac{\nu}{\sqrt{1-\nu^2}} \right]. \quad (3.11)$$

### The Legendre transformation

It would be worth mentioning about an additional contribution coming from the boundary [35]. In the undeformed case, it is well recognized that the classical action has a linear divergence and then it can be removed by considering a Legendre transformation. The origin of this additional contribution is the surface term which appears in taking a variation of the classical action to obtain the equations motion. This is just because the minimal surface has the boundary. Hence one needs to discuss this contribution in the deformed case as well. By taking account of a Legendre transformation, i.e., adding a total derivative to the Lagrangian, the total action is written as

$$\tilde{S} = S + S_L, \quad S_L = \int_0^{2\pi} d\tau \int_{\sigma_0}^{\infty} d\sigma \partial_{\sigma} \left( z \frac{\partial L}{\partial(\partial_{\sigma} z)} \right). \quad (3.12)$$

The term  $S_L$  is a total derivative and it is evaluated as

$$\begin{aligned} S_L &= - \int_0^{2\pi} d\tau z \frac{\partial L}{\partial(\partial_{\sigma} z)} \Big|_{\sigma=\sigma_0} = -\sqrt{\lambda} \frac{\sqrt{1+C^2} \tanh \sigma_0}{\sinh^2 \sigma_0 + C^2 \cosh^2 \sigma_0} \\ &= -\sqrt{\lambda} \sqrt{1+C^2} \frac{\epsilon(a^2 - \epsilon^2)}{a(\epsilon^2 + C^2 a^2)}. \end{aligned} \quad (3.13)$$

A remarkable point is that (3.13) vanishes in the limit  $\epsilon \rightarrow 0$  when  $C \neq 0$ , and it does not contribute to the final expression of the action.

The next is to consider the undeformed limit of  $S_L$  so as to cancel the divergent term in the undeformed limit of  $S$  (3.5). The term  $S_L$  in (3.13) vanishes when  $C \neq 0$ , hence this is the first place one need to introduce  $\epsilon$  so as to keep the boundary away from the singularity surface. By taking the  $C \rightarrow 0$  limit with  $\epsilon$  fixed, the resulting expression of  $S_L$  is given by

$$S_L = -\sqrt{\lambda} \frac{a}{\epsilon} + \mathcal{O}(\epsilon), \quad (3.14)$$

and it cancels the divergent term in (3.9) as usual.

It would be helpful to comment on the cut-off  $\epsilon$  and the deformation parameter  $C$ . When  $C \neq 0$ , there is no strict need to introduce  $\epsilon$  in evaluating the classical action  $S$  and the Legendre term  $S_L$  vanishes. However, if the limit  $\epsilon \rightarrow 0$  is taken first, or  $\epsilon$  is not turned on, the finite result cannot be reproduced correctly in the  $C \rightarrow 0$  limit. In this scene, there is a subtlety of the order the two limits: 1)  $\epsilon \rightarrow 0$  and 2)  $C \rightarrow 0$ . At least so far, a possible resolution is to take the limit  $C \rightarrow 0$  first while keeping  $\epsilon$  finite. In the following, this order of the limits is supposed.

## A multiple string case

Finally, let us comment on a generalization of the solution (3.3) with wrapping  $k$  times around the circle. The solution is described by

$$z = a \tanh k\sigma, \quad r = \frac{a}{\cosh k\sigma}, \quad \psi_1 = k\tau. \quad (3.15)$$

By introducing a cut-off  $\epsilon$  on  $z$  through

$$\epsilon = a \tanh k\sigma_0, \quad (3.16)$$

then the classical action can be evaluated as

$$S = \sqrt{\lambda} \frac{\sqrt{1+C^2}}{C} k \left( \operatorname{arccot}[C] - \operatorname{arccot}\left[\frac{Ca}{\epsilon}\right] \right). \quad (3.17)$$

The second term in (3.17) vanishes in the  $\epsilon \rightarrow 0$  limit with non-vanishing  $C$ . Hence, when  $C \neq 0$ , no UV cut-off has to be introduced as in the previous case. In the  $C \rightarrow 0$  limit with non-zero  $\epsilon$ , the classical action (3.17) reproduces the undeformed result [36],

$$S = -\sqrt{\lambda} k + \sqrt{\lambda} \frac{ak}{\epsilon}. \quad (3.18)$$

As for the contribution of the Legendre transformation, when  $C \neq 0$ , it vanishes in the  $\epsilon \rightarrow 0$  limit as well. In the undeformed limit  $C \rightarrow 0$  with  $\epsilon$  fixed, it cancels the divergent term in (3.18).

### 3.2 A deformed $\text{AdS}_2 \times \text{S}^2$ subspace

The next issue is to consider a generalization of the solution constructed in the previous subsection. As we will show below, the solution is reduced to a string solution dual to a  $1/4$  BPS circular Wilson loop [36] in the undeformed limit.

Firstly, let us suppose the following ansatz with the conformal gauge :

$$\begin{aligned} z &= \sqrt{a^2 - r^2}, & r &= r(\sigma), & \psi_1 &= \psi(\tau), & \psi_2 &= \zeta = 0, \\ \gamma &= \gamma(\sigma), & \phi_1 &= \phi(\tau), & \phi &= \phi_2 = \xi = 0, \end{aligned} \quad (3.19)$$

with  $0 \leq \tau < 2\pi$  and  $0 \leq \sigma < \infty$ . Here  $a$  is the radius of the circle at the boundary. Note that (3.19) is a consistent ansatz and the WZ term vanishes. Then the deformed  $\text{AdS}_5 \times \text{S}^5$  is reduced to a deformed  $\text{AdS}_2 \times \text{S}^2$  with the metric

$$ds_{\text{AdS}_2 \times \text{S}^2}^2 = R^2 \sqrt{1 + C^2} \left[ \frac{r^2}{(1 + C^2)a^2 - r^2} \left[ \frac{a^2 dr^2}{r^2(a^2 - r^2)} + d\psi_1^2 \right] + \frac{d\gamma^2 + \sin^2 \gamma d\phi_1^2}{1 + C^2 \cos^2 \gamma} \right].$$

Here let us introduce a latitude angle  $\theta_0$  for the sphere part, and the boundary condition is set to

$$\gamma(\sigma = 0) = \frac{\pi}{2} - \theta_0. \quad (3.20)$$

When  $\theta_0 = \pi/2$ , the surface in the sphere part is reduced to a point (the north pole). The resulting solution is just the deformed  $\text{AdS}_2$  one discussed in section 3.1.

Then the classical solution is given by

$$\begin{aligned} z &= a \tanh \sigma, & r &= \frac{a}{\cosh \sigma}, & \sin \gamma &= \frac{1}{\cosh(\zeta_0 \pm \sigma)}, \\ \psi_1 &= \tau, & \phi_1 &= \tau. \end{aligned} \quad (3.21)$$

Here the signatures  $\pm$  in  $\sin \gamma$  correspond to the surface which extends over the northern (the short side) or southern hemisphere. An integration constant  $\zeta_0$  can be fixed by the boundary condition at  $\sigma = 0$ ,

$$\sin \gamma(\sigma = 0) = \frac{1}{\cosh \zeta_0} = \cos \theta_0. \quad (3.22)$$

The remaining task is to evaluate the classical Euclidean action. Let us formally introduce a cut-off  $\epsilon$  for  $z$ . Then it can be converted into a cut-off for  $\sigma$  at  $\sigma_0$  through the classical solution (3.21) for  $z$ ,

$$\epsilon = a \tanh \sigma_0. \quad (3.23)$$

Then the classical action is given by

$$\begin{aligned}
S &= \frac{\sqrt{\lambda}}{4\pi} \sqrt{1+C^2} \int_0^{2\pi} d\tau \int_{\sigma_0}^{\infty} d\sigma \left[ \frac{2}{\sinh^2 \sigma + C^2 \cosh^2 \sigma} + \frac{2}{\cosh^2(\varsigma_0 \pm \sigma) + C^2 \sinh^2(\varsigma_0 \pm \sigma)} \right] \\
&= \sqrt{\lambda} \frac{\sqrt{1+C^2}}{C} \left( \operatorname{arccot}[C] - \operatorname{arccot}\left[\frac{Ca}{\epsilon}\right] \right. \\
&\quad \left. + \arctan[C] \mp \arctan\left[C \tanh\left(\operatorname{arccosh}[\sec \theta_0] \pm \operatorname{arctanh}\left[\frac{\epsilon}{a}\right]\right)\right] \right). \quad (3.24)
\end{aligned}$$

The action (3.24) can be expanded in terms of  $\epsilon$  and there is no divergence with non-vanishing  $C$ . By taking the  $\epsilon \rightarrow 0$  limit ( $C$ : fixed), the action (3.24) becomes

$$S = \sqrt{\lambda} \frac{\sqrt{1+C^2}}{C} \left( \operatorname{arccot}[C] + \arctan[C] \mp \arctan[C \sin \theta_0] \right). \quad (3.25)$$

Thus the expression (3.25) is finite even if there is no UV cut-off for the string world-sheet. It indicates again that the parameter  $C$  plays a role as a UV regularization. Note that  $C$  can be converted into the  $q$ -deformation parameter through the relation (3.10). Then the result (3.25) is expressed as

$$S = \frac{\sqrt{\lambda}}{\nu} \left( \operatorname{arccot}\left[\frac{\nu}{\sqrt{1-\nu^2}}\right] + \arctan\left[\frac{\nu}{\sqrt{1-\nu^2}}\right] \mp \arctan\left[\frac{\nu}{\sqrt{1-\nu^2}} \sin \theta_0\right] \right). \quad (3.26)$$

It is worth seeing the undeformed limit. By taking the  $C \rightarrow 0$  limit, the classical action (3.24) is reduced to

$$S = \sqrt{\lambda} \left( \frac{a}{\epsilon} \mp \sin \theta_0 \right) + \mathcal{O}(\epsilon). \quad (3.27)$$

After subtracting the divergent term, the undeformed result [36] is precisely reproduced.

As for the boundary contribution, there is no contribution of  $S_L$  because it vanishes in the  $\epsilon \rightarrow 0$  limit when  $C \neq 0$ , as in Sec. 3.1. Similarly,  $S_L$  cancels the divergent term in the  $C \rightarrow 0$  limit with  $\epsilon$  fixed.

## A multiple string case

Finally, let us comment on a generalization of the solution (3.21) with wrapping  $k$  and  $m$  times around the circle in the deformed  $\text{AdS}_2$  and the deformed  $\text{S}^2$ , respectively. The classical solution is given by

$$z = a \tanh k\sigma, \quad r = \frac{a}{\cosh k\sigma}, \quad \sin \gamma = \frac{1}{\cosh m(\varsigma_0 \pm \sigma)},$$

$$\psi_1 = k\tau, \quad \phi_1 = m\tau. \quad (3.28)$$

The integration constant  $\varsigma_0$  is fixed again by the boundary condition at  $\sigma = 0$ ,

$$\sin \gamma(\sigma = 0) = \frac{1}{\cosh m\varsigma_0} = \cos \theta_0. \quad (3.29)$$

By formally introducing a cut-off  $\epsilon$  on  $z$  through

$$\epsilon = a \tanh k\sigma_0, \quad (3.30)$$

then the classical action can be evaluated as

$$\begin{aligned} S &= \frac{\sqrt{\lambda}}{4\pi} \sqrt{1+C^2} \int_0^{2\pi} d\tau \int_{\sigma_0}^{\infty} d\sigma \left[ \frac{2}{\sinh^2 k\sigma + C^2 \cosh^2 k\sigma} \right. \\ &\quad \left. + \frac{2}{\cosh^2 m(\varsigma_0 \pm \sigma) + C^2 \sinh^2 m(\varsigma_0 \pm \sigma)} \right] \\ &= \sqrt{\lambda} \frac{\sqrt{1+C^2}}{C} \left[ k \left( \operatorname{arccot}[C] - \operatorname{arccot}\left[\frac{Ca}{\epsilon}\right] \right) \right. \\ &\quad \left. + m \left( \arctan[C] \mp \arctan \left[ C \tanh \left( \operatorname{arccosh}[\sec \theta_0] \pm \frac{m}{k} \operatorname{arctanh} \left[ \frac{\epsilon}{a} \right] \right) \right] \right) \right]. \quad (3.31) \end{aligned}$$

In the  $\epsilon \rightarrow 0$  limit with non-vanishing  $C$ , the action (3.31) becomes

$$S = \sqrt{\lambda} \frac{\sqrt{1+C^2}}{C} \left( k \operatorname{arccot}[C] + m \left( \arctan[C] \mp \arctan[C \sin \theta_0] \right) \right). \quad (3.32)$$

By taking the  $C \rightarrow 0$  limit with non-zero  $\epsilon$ , the classical action (3.31) reproduces to the undeformed result [36],

$$S = \sqrt{\lambda} \left( -k + k \frac{a}{\epsilon} + m \mp m \sin \theta_0 \right) + \mathcal{O}(\epsilon). \quad (3.33)$$

As for the contribution of the Legendre transformation, when  $C \neq 0$ , it vanishes in the  $\epsilon \rightarrow 0$  limit as well. In the undeformed limit  $C \rightarrow 0$  with  $\epsilon$  fixed, it cancels the divergent term in (3.33).

## 4 A minimal surface with a cylinder shape

Due to the deformation, it seems difficult to construct a straight string solution with the Poincare coordinates, while it is possible to construct a cylinder shape solution.

Hence, in this section, we shall consider a minimal surface solution with a cylinder shape. Then the solution ends with a circle on the boundary. So, with the conformal gauge, we are concerned with the following ansatz:

$$z = z(\sigma), \quad r = a, \quad \psi_1 = \psi_1(\tau), \quad \psi_2 = \zeta = 0, \quad (4.1)$$

with  $0 \leq \tau < 2\pi$ ,  $0 \leq \sigma < \infty$ . Here  $a$  is a positive real constant. Note that (4.1) is a consistent ansatz and the WZ term vanishes. Then the geometry associated with the solution is described by a deformed  $\text{AdS}_2$  with the metric

$$ds_{\text{AdS}_2}^2 = \frac{R^2 \sqrt{1+C^2}}{(1+C^2)z^2 + C^2 a^2} \left[ (1+C^2) dz^2 + a^2 d\psi_1^2 \right]. \quad (4.2)$$

This metric leads to a solution of the string equation of motion :

$$z = \frac{a \sigma}{\sqrt{1+C^2}}, \quad \psi_1 = \tau. \quad (4.3)$$

The next task is to evaluate the classical Euclidean action for this solution. Let us formally introduce a cut-off  $\epsilon$  for  $z$ . Then it can be converted into a cut-off for  $\sigma$  at  $\sigma_0$  via the classical solution (4.3) for  $z$ ,

$$\epsilon = \frac{a \sigma_0}{\sqrt{1+C^2}}. \quad (4.4)$$

Then the classical action is evaluated as

$$\begin{aligned} S &= \frac{\sqrt{\lambda}}{4\pi} \sqrt{1+C^2} \int_0^{2\pi} d\tau \int_{\sigma_0}^{\infty} d\sigma \frac{2}{\sigma^2 + C^2} \\ &= \sqrt{\lambda} \frac{\sqrt{1+C^2}}{C} \arctan \left[ \frac{C}{\sqrt{1+C^2}} \frac{a}{\epsilon} \right]. \end{aligned} \quad (4.5)$$

The action (4.5) can be expanded in terms of  $\epsilon$  and there is no divergent term with non-zero  $C$ . By taking  $\epsilon \rightarrow 0$ , the classical action (4.5) becomes,

$$S = \sqrt{\lambda} \frac{\sqrt{1+C^2}}{C} \frac{\pi}{2}. \quad (4.6)$$

The action (4.6) is finite and hence no UV cut-off is needed for the string world-sheet. It is remarkable again that  $C$  plays a role of UV regularization. Note that  $C$  can be converted into the  $q$ -deformation parameter via the relation (3.10), and then (4.6) is expressed as

$$S = \frac{\sqrt{\lambda}}{\nu} \frac{\pi}{2}. \quad (4.7)$$

Let us comment on the undeformed limit of the classical action (4.5). By taking the  $C \rightarrow 0$  limit ( $\epsilon$ : fixed), the undeformed result mentioned in [34] is reproduced as

$$S = \sqrt{\lambda} \frac{a}{\epsilon}. \quad (4.8)$$

This divergent term is canceled out with the boundary contribution as we will see below.

It is of importance to consider the boundary term via a Legendre transformation [35]. In the present case, the total derivative term is given by

$$S_L = - \int_0^{2\pi} d\tau z \frac{\partial L}{\partial(\partial_\sigma z)} \Big|_{\sigma=\sigma_0} = -\sqrt{\lambda} \frac{(1+C^2) a \epsilon}{\epsilon^2 + C^2(\epsilon^2 + a^2)}. \quad (4.9)$$

When  $C \neq 0$ ,  $S_L$  vanishes in the  $\epsilon \rightarrow 0$  limit and so it does not contribute to the final expression. On the other hand, in the undeformed limit  $C \rightarrow 0$  limit ( $\epsilon$ : fixed),  $S_L$  is reduced to the following form,

$$S_L = -\sqrt{\lambda} \frac{a}{\epsilon}, \quad (4.10)$$

and it cancels the divergent term in (4.8).

## 5 Conclusion and discussion

In this letter, we have discussed a  $q$ -deformation of the  $\text{AdS}_5 \times \text{S}^5$  superstring. It is conjectured in [21] that a singularity surface in the ABF metric may be regarded as a holographic screen in the deformed theory. To look for some support for this conjecture, we have further considered minimal surfaces whose boundary shape is a circle, by using the associated Poincare coordinates. A remarkable feature is that the classical Euclidean action is not divergent unlike the undeformed one. The finiteness comes from the fact that the  $q$ -deformation may be regarded as a UV regularization. As a matter of course, the divergent term is reproduced in the undeformed limit. That is, the undeformed limit becomes singular. This result may indicate that our conjecture would make sense. To gain more support, it would be interesting to consider other minimal surfaces like cusp Wilson loop by employing the Poincare coordinates.

There exist many open problems. As a matter of course, the most interesting issue is to unveil the gauge-theory side dual to the  $q$ -deformation of the  $\text{AdS}_5 \times \text{S}^5$  superstring.

To tackle it, the first thing we have to do is to understand the boundary geometry of the deformed  $\text{AdS}_5$  with the Poincare coordinates. Naively, the boundary geometry degenerates. Let us consider the metric (2.9) and naively study the boundary geometry by following the usual prescription. For the finiteness of the boundary metric, the scale factor  $\Omega^2 \equiv z^2(z^2 + C^2(z^2 + r^2))$  is multiplied to the metric (2.9). Then, after taking the  $z \rightarrow 0$  limit, the resulting metric is dimensionally reduced to  $ds^2 = R^2\sqrt{1+C^2}C^2r^2dr^2$ . This is not intrinsic to our scenario but a general problem which typically appears when considering non-asymptotically AdS spaces including the  $q$ -deformed AdS space. It may be interesting to try to resolve this issue by following [37], but we have no idea for its resolution so far. Probably, it would be nice to consider an appropriate scaling limit of the deformed Poincare  $\text{AdS}_5$ . We hope we could report the result in the near future.

To identify the dual gauge-theory side, our result may be a key ingredient. One plausible speculative scenario is the following. Firstly, deduce a candidate of the dual gauge-theory from the UV finiteness. Then one can compute the vacuum expectation value of a circular Wilson loop in the field-theory side by summing up rainbow diagrams [38] or using a localization technique [39]. As a result, the strong-coupling result can be extracted. If the result on the gauge-theory side agrees with the one we obtained in the string-theory side, we would get support for the candidate gauge theory.

We believe that our results on minimal surfaces can play an important role in uncovering a possible gauge-theory dual.

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# Appendix

## A Proper distance to the singularity surface

Here we consider space-like paths to the singularity surface in the global deformed  $\text{AdS}_5$  (2.1) with the Lorentzian signature and their proper distances.

Let us focus on a radial space-like geodesic  $x = (t(t_A), \chi(t_A))$  which satisfies a unit speed parametrization,

$$\dot{x}^2 = -\cosh^2 \chi \dot{t}^2 + \frac{\dot{\chi}^2}{1 + C^2 \cosh^2 \chi} = +1, \quad (\text{A.1})$$

where  $t_A$  is an affine parameter and  $\dot{x} = dx/dt_A$ . For simplicity, we have omitted the overall factor  $R^2\sqrt{1+C^2}$ . The path can be solved by extremizing the action,

$$S = \int dt_A \left[ -\cosh^2 \chi \dot{t}^2 + \frac{\dot{\chi}^2}{1 + C^2 \cosh^2 \chi} \right]. \quad (\text{A.2})$$

As a result, we obtain the following expressions:

$$E = \cosh^2 \chi \dot{t}, \quad \dot{\chi}^2 = (1 + C^2 \cosh^2 \chi) \frac{E^2 + \cosh^2 \chi}{\cosh^2 \chi}. \quad (\text{A.3})$$

Here  $E$  is a conserved quantity (global AdS energy) defined as  $E \equiv -p_t$  with  $p_t \equiv G_{tt} dt/dt_A$ .

Then the proper distance from the origin to the singularity surface is given by

$$\begin{aligned} t_A &= \int_0^\infty \frac{d\chi}{\dot{\chi}} = \int_0^\infty \frac{d\chi \cosh \chi}{\sqrt{(1 + C^2 \cosh^2 \chi)(E^2 + \cosh^2 \chi)}} \\ &= \frac{1}{\sqrt{1 + C^2}} \text{K} \left[ \frac{1 - C^2 E^2}{1 + C^2} \right]. \end{aligned} \quad (\text{A.4})$$

This result indicates that in the deformed case the proper distance to the singularity surface is finite for any  $C$ , while the space-like distance in the usual  $\text{AdS}_5$  is infinite. The purely radial case is obtained by taking  $E = 0$  giving

$$t_A = \frac{1}{\sqrt{1 + C^2}} \text{K} \left[ \frac{1}{1 + C^2} \right]. \quad (\text{A.5})$$

The causal physical quantities in the Lorentzian signature are irrelevant to this result. However, in the Euclidean signature, this behavior of the space-like proper distance is responsible and would explain that the minimal surface areas are finite even if no UV cut-off has been introduced.

## B Poincare coordinates for the ABF metric

We shall introduce the Poincare coordinates for the ABF metric [13].

The ABF metric for the AdS part is given by

$$ds_{\text{AdS}_5}^2 = R^2(1+C^2)^{\frac{1}{2}} \left[ -\frac{\cosh^2 \rho dt^2}{1-C^2 \sinh^2 \rho} + \frac{d\rho^2}{1-C^2 \sinh^2 \rho} + \frac{\sinh^2 \rho d\zeta^2}{1+C^2 \sinh^4 \rho \sin^2 \zeta} \right. \\ \left. + \frac{\sinh^2 \rho \cos^2 \zeta d\psi_1^2}{1+C^2 \sinh^4 \rho \sin^2 \zeta} + \sinh^2 \rho \sin^2 \zeta d\psi_2^2 \right]. \quad (\text{B.1})$$

Here the deformed  $\text{AdS}_5$  is parameterized by the coordinates  $(t, \psi_1, \psi_2, \zeta, \rho)$ . The singularity surface exists at  $\rho = \text{arcsinh}(1/C)$ . The deformation is measured by a real parameter  $C \in [0, \infty)$ .

Here we consider the Poincare coordinate for the deformed  $\text{AdS}_5$  (B.1). It is helpful to take a coordinate transformation

$$\cosh \rho \equiv \frac{1}{\cos \vartheta}, \quad (\text{B.2})$$

and perform the Wick rotation  $t \rightarrow i\tau$ .

Then, performing a coordinate transformation

$$z \equiv e^\tau \cos \vartheta, \quad r \equiv e^\tau \sin \vartheta, \quad (\text{B.3})$$

the metric (B.1) can be rewritten into the Poincare form,

$$ds_{\text{AdS}_5}^2 = R^2 \sqrt{1+C^2} \left[ \frac{dz^2 + dr^2}{z^2 - C^2 r^2} + \frac{r^2(d\zeta^2 + \cos^2 \zeta d\psi_1^2)}{z^2 + C^2 (\frac{r}{z})^2 r^2 \sin^2 \zeta} + \frac{r^2 \sin^2 \zeta d\psi_2^2}{z^2} \right]. \quad (\text{B.4})$$

By taking the following coordinate transformation:

$$\sqrt{1+C^2} z \equiv \sqrt{z^2 - C^2 r^2}, \quad r \equiv r, \quad (\text{B.5})$$

the resulting metric corresponds to the Poincare coordinates (2.9).

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